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ANALYTIC FACTORIZATIONS AND COMPLETELY BOUNDED MAPS

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ABSTRACT

We prove an analytic factorization theorem in the setting of the recently developed theory of operator spaces. We especially obtain the following result: Let A be a C^*-algebra and H be a Hilbert space. Let φ be an element of $H^\infty(CB(A, B(H)))$, i.e. a bounded analytic function valued in the space of completely bounded maps from A into $B(H)$. Then there exist a Hilbert space K, a representation $\pi: A \to B(K), \varphi_1 \in H^\infty(B(H,K))$ and $\varphi_2 \in H^\infty(B(K,H))$ such that $\|\varphi_1\|_\infty \, \|\varphi_2\|_\infty \leq \|\varphi\|_\infty$ and:

$$\forall z \in D,\ \forall a \in A, \quad \varphi(z)(a) = \varphi_2(z)\pi(a)\varphi_1(z).$$

We also prove an analogous result for completely bounded multilinear maps. The last part of the paper is devoted to a new proof of Pisier's theorem about gamma-norms.

1. Introduction

A few years ago, Haagerup and Pisier [HP] proved the following analytic factorization theorem:

Let A be a C^*-algebra and let $\varphi \in H^\infty(A^*)$. Then there are a Hilbert space K, a representation $\pi: A \to B(K)$, $\varphi_1 \in H^\infty(K)$ and $\varphi_2 \in H^\infty(K^*)$ such that $\|\varphi_1\|_\infty \|\varphi_2\|_\infty \leq \|\varphi\|_\infty$ and:

$$\forall\ a \in A,\ \forall\ z \in D \quad (\varphi(z))(a) = \langle \pi(a)\varphi_1(z), \varphi_2(z)\rangle.$$

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The purpose of this paper is to study various generalizations of this result in the setting of completely bounded multilinear maps. One of the most striking results in the theory of completely bounded maps is the representation theorem by Christensen and Sinclair [CS1]. Our first motivation was to obtain an analytic form of this. In section 4, we will prove the following:

THEOREM 1.1: *Let $A_1, \ldots, A_n$ be C^*-algebras. Let E, F be Hilbert spaces. Let $\varphi \in H^\infty(CB(A_n \times \cdots \times A_1, B(E,F)))$. Then there exist Hilbert spaces K_i $(1 \leq i \leq n)$, representations π_i: $A_i \to B(K_i)$ and analytic functions $\varphi_0 \in H^\infty(B(E,K_1))$, $\varphi_n \in H^\infty(B(K_n,F))$, $\varphi_i \in H^\infty(B(K_i,K_{i+1}))$ $(1 \leq i \leq n-1)$ such that:*

(i) $\|\varphi_0\|_\infty \, \|\varphi_1\|_\infty \cdots \|\varphi_n\|_\infty \leq \|\varphi\|_\infty$,

(ii) $\forall \ (a_n, \ldots, a_1) \in A_n \times \cdots \times A_1$, $\forall \ z \in D$:

$$(\varphi(z))(a_n, \ldots, a_1) = \varphi_n(z)\pi_n(a_n)\varphi_{n-1}(z) \ldots \ldots \varphi_1(z)\pi_1(a_1)\varphi_0(z).$$

Before going further, let us explain the notation in the preceding statement.

We let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc. Let X be a complex Banach space. We denote by $H^\infty(X)$ the space of all bounded functions f: $D \to X$ equipped with the (complete) norm: $\|f\|_\infty = \mathrm{Sup}_{z \in D} \|f(z)\|_X$.

As usual, we denote by $B(E,F)$ the space of all bounded linear maps from E into F. The notation $CB(A_n \times \cdots \times A_1, B(E,F))$ stands for the space of completely bounded multilinear maps from $A_n \times \cdots \times A_1$ into $B(E,F)$. The necessary definitions will be given below. For a wide information about completely bounded maps, we refer the reader to [Pa] and [CS2].

Along this paper, we will frequently use results and ideas from the theroy of operator spaces which was recently developped by Blecher–Paulsen [BP], [B1], [B2], Effros–Ruan [ER1], [ER2] and others. We will especially use the link between the representation of completely bounded maps and Hilbert space factorizations (see [ER2], [B2]).

In section 2, we review some definitions and results about operator spaces which will be used further and prove simple results about Hilbert space factorizations.

In section 3, we state and prove our main result from which Theorem 1.1 will be deduced in section 4. The remainder of this fourth section is devoted to some corollaries of Theorem 1.1 which are inspired by results from [HP]. For example, we obtain that given two Hilbert spaces E, H, and a C^*-algebra $A \subset B(H)$, any bounded analytic function valued in $CB(A, B(E))$ can be extended to a bounded analytic function valued in $CB(B(H), B(E))$.

In section 5, we are interested in gamma-norms. These norms were introduced in [Pi1] where Pisier especially proved various generalizations of Sarason's factorization theorem [S]. New developments about these norms may be found in [Pi4]. Many of the results in [Pi1] are based on an analytic factorization theorem for gamma-norms. We give a new proof of this theorem, stated below as Theorem 5.1. It should be noticed that using Pisier's Theorem 5.1, it is not hard to show that the proof of our Theorem 1.1 reduces to the particular case when $n = 1$ (see Remark 5.5). To emphasize the difference between our approach and Pisier's one, we notice that the originate proof of Theorem 5.1 heavily relies upon the lifting of the commutant theorem due to Nagy and Foias [NF]. Conversely, Sarason's factorization theorem (which is known to be the dual formulation of the Nagy–Foias theorem) appears as a corollary in our work (see Remark 4.7).

We end this introduction by a warning: along this paper, we will not use the natural identification between a Hilbert space H and its dual H^*. For any $T\colon H \to K$, we denote by $T^*\colon K^* \to H^*$ its transposed map.

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2. Preliminaries about operator spaces

We first recall some basic definitions about operator spaces. We follow [BP],[B1], [ER1], [ER2]. Let X be a complex Banach space. Let us assume that for any $n \geq 1$, we are given a norm on the matrix space $\mathcal{M}_n(X)$. For any $x \in \mathcal{M}_n(X)$ and $y \in \mathcal{M}_m(X)$, we let $x \oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in \mathcal{M}_{n+m}(X)$. Then we consider two properties about the matrix norms on X:

$$\forall x \in \mathcal{M}_n(X), \forall\, y \in \mathcal{M}_m(X), \quad \|x \oplus y\| = \max\{\|x\|, \|y\|\}, \tag{2.1}$$

$$\forall \alpha \in \mathcal{M}_{nm}, \forall\, x \in \mathcal{M}_m(X), \forall \beta \in \mathcal{M}_{mn}, \quad \|\alpha x \beta\| \leq \|\alpha\|\, \|x\|\, \|\beta\|. \tag{2.2}$$

Ruan's representation theorem [R, ER3] states that X satisfies (2.1) and (2.2) iff there is an embedding $X \subset B(H)$ for some Hilbert space H such that:

$$\forall x = [x_{ij}] \in \mathcal{M}_n(X),$$
$$\|x\| = \text{ Sup } \Big\{ \Big(\sum_{i=1}^n \|\sum_{j=1}^n x_{ij}(h_j)\|^2 \Big)^{1/2} \Big| \, h_1, \dots, h_n \in H, \ \sum_{j=1}^n \|h_j\|^2 \leq 1 \Big\}.$$

This means that all the embeddings $\mathcal{M}_n(X) \subset B(\ell_n^2(H))$ are isometries. In such a case, X is called an operator space. The norms on the spaces $\mathcal{M}_n(X)$ determine unique norms on rectangular matrices spaces $\mathcal{M}_{pq}(X)$ obtained by the natural inclusion of any $\mathcal{M}_{pq}(X)$ in a square matrices space. It is well-known that for a C^*-algebra A, all the C^*-algebraic embeddings $A \subset B(H)$ provide the same norms on the spaces $\mathcal{M}_n(A)$. Therefore, given a C^*-algebra A, we will always assume that it is endowed with this intrinsic system of matrix norms. More generally, given two Hilbert spaces E, F, we will assume that $B(E,F)$ is endowed with the matrix norms defined by the identifications

$$\mathcal{M}_n(B(E,F)) = B(\ell_n^2(E), \ell_n^2(F)).$$

Let X, Y be two operator spaces. Let $u \in B(X,Y)$. We define $u^{(n)}\colon \mathcal{M}_n(X) \to \mathcal{M}_n(Y)$ by $u^{(n)}([x_{ij}]) = [u(x_{ij})]$. We let $\|u\|_{cb} = \text{Sup}_{n \geq 1} \|u^{(n)}\|$. We say that u is completely bounded (in short c.b.) provided that $\|u\|_{cb} < +\infty$. We denote by $CB(X,Y)$ the resulting Banach space. Let $u \in CB(X,Y)$. We say that u is completely contractive (in short c.c.) provided that $\|u\|_{cb} \leq 1$. We say that u is completely isometric provided that for any $n \geq 1$, $u^{(n)}$ is an isometry. A completely isometric isomorphism $u \in CB(X,Y)$ allows us to identify X and Y as operator spaces. Such an identification will be denoted by "$X \cong Y$". To avoid confusion, we will keep the symbol "=" to mention a Banach spaces identification.

We now turn to the Haagerup tensor product. Let X, Y be operator spaces. Given $x = [x_{ik}] \in \mathcal{M}_{np}(X)$ and $y = [y_{kj}] \in \mathcal{M}_{pn}(Y)$, we define $x \odot y \in \mathcal{M}_n(X \otimes Y)$ by $x \odot y = [\sum_{k=1}^p x_{ik} \otimes y_{kj}]$. The space $X \overset{h}{\otimes} Y$ is defined as the completion of $X \otimes Y$ equipped with the following matrix norms. For any $n \geq 1$ and any $v \in \mathcal{M}_n(X \otimes Y)$, we let:

$$\|v\| = \inf \{ \|x\| \, \|y\| \ | \ x \in \mathcal{M}_{np}(X), \ y \in \mathcal{M}_{pn}(Y), \ v = x \odot y \}.$$

These norms satisfy (2.1) and (2.2). Therefore, by Ruan's theorem, $X \overset{h}{\otimes} Y$ is an operator space. It is called the Haagerup tensor product of X and Y. We recall that the Haagerup norm is injective, but it is not commutative. Moreover it is associative. This allows us to consider the Haagerup tensor product $X_n \overset{h}{\otimes} \cdots \overset{h}{\otimes} X_1$ of a finite family $(X_n, \ldots, X_1)$ of operator spaces.

Let $X_1, \ldots, X_n, Y$ be operator spaces. Let φ: $X_n \times \cdots \times X_1 \to Y$ be a bounded multilinear map. It may be viewed as a linear map from $X_n \otimes \cdots \otimes X_1$ to Y. We say that φ is completely bounded if and only if it defines a completely bounded map $\widehat{\varphi}$: $X_n \overset{h}{\otimes} \cdots \overset{h}{\otimes} X_1 \to Y$. We then let $\|\varphi\|_{cb} = \|\widehat{\varphi}\|_{cb}$. We denote by $CB(X_n \times \cdots \times X_1, Y)$ the resulting Banach space and of course $CB(X_n \times \cdots \times X_1, Y) = CB(X_n \overset{h}{\otimes} \cdots \overset{h}{\otimes} X_1, Y)$. The multilinear analogue of the Wittstock–Stinespring representation theorem for completely bounded maps is the following theorem by Paulsen and Smith:

THEOREM 2.1 ([PS, p. 272]): *Let $X_i \subset B(H_i)$ $(1 \leq i \leq n)$ be operator spaces. Let E, F be Hilbert spaces. Let φ: $X_n \times \cdots \times X_1 \to B(E, F)$ be a completely bounded multilinear map with $\|\varphi\|_{cb} \leq 1$.*

Then there exist Hilbert spaces $K_i (1 \leq i \leq n)$, representations π_i: $B(H_i) \to B(K_i)(1 \leq i \leq n)$ and contractions T_0: $E \to K_1, T_n$: $K_n \to F$ and T_i: $K_i \to K_{i+1}(1 \leq i \leq n-1)$ such that: $\varphi(x_n, \ldots, x_1) = T_n \pi_n(x_n)\, T_{n-1} \ldots T_1\, \pi_1(x_1) T_0$.

In the particular case when $X_1, \ldots, X_n$ are C^*-algebras, Theorem 2.1 had been previously proved by Christensen and Sinclair [CS1]. Thus, as mentioned in the introduction, Theorem 1.1 may be viewed as an analytic version of Christensen and Sinclair's theorem.

Let us now point out some more operator spaces that will be used in this paper. Let X, Y be operator spaces. Let $u = [u_{ij}] \in \mathcal{M}_n(CB(X, Y))$. We can regard u as a map from X to $\mathcal{M}_n(Y)$ by letting $u(x) = [u_{ij}(x)]$. The resulting identification

$$\mathcal{M}_n(CB(X, Y)) = CB(X, \mathcal{M}_n(Y))$$

defines a norm on $\mathcal{M}_n(CB(X, Y))$. These norms satisfy (2.1) and (2.2). Therefore, when endowed with these matrix norms, $CB(X, Y)$ is an operator space.

In the particular case $Y = \mathbb{C}$, we get X^* as an operator space. Namely, we have $\mathcal{M}_n(X^*) = CB(X, \mathcal{M}_n)$. When it is equipped with this operator space structure, X^* is called the standard dual of X (see [B1] for a complete information). The standard duality of operator spaces is very useful since we have:

$$\forall u \in B(X, Y), \qquad \|u\|_{cb} = \|u^*\|_{cb} \tag{2.3}$$

(see [ER1, th 2.2] or [BP, th 2.11]).

Let H be a Hilbert space. Following [ER2], we denote by H_c the operator space structure on H given by the identification $H_c \cong B(\mathbb{C}, H)$. It is called the column operator structure on H. Similarly, we denote by H_r the so-called row operator structure on H given by $H_r \cong B(H^*, \mathbb{C})$. The notation H_r^* will stand for $(H^*)_r$, i.e. $B(H, \mathbb{C})$.

We are now ready to state an identification result which will be of crucial use in the sequel.

Let G be an operator space, let E, F be Hilbert spaces. Let $u \in CB(G, B(E, F))$. We can regard u as a trilinear form $\hat{u}$ on $F^* \times G \times E$ by letting $\hat{u}(f^*, g, e) = \langle u(g)(e), f^* \rangle$. The map $u \mapsto \hat{u}$ gives rise to:

$$CB(G, B(E, F)) \cong \big(F_r^* \overset{h}{\otimes} G \overset{h}{\otimes} E_c\big)^* \tag{2.4}$$

(see [ER2, cor 4.6]).

In particular ($E = \mathbb{C}$), we have for any operator space G and for any Hilbert space F:

$$\big(F_r^* \overset{h}{\otimes} G\big)^* \cong CB(G, F_c). \tag{2.5}$$

In the very particular case when $G = \mathbb{C}$, we get for any Hilbert space F:

$$(F_r^*)^* \cong F_c. \tag{2.6}$$

The last part of this section is devoted to connections between completely bounded multilinear maps and Hilbert space factorizations. This subject was especially studied in [ER2, Section 5] (see also [B2] and [BP]). Using Effros and Ruan's method, we can get the following:

Proposition 2.2: *Let V, W be two operator spaces. Let $A \subset B(E)$ be a third operator space and let T: $W \times A \times V \to \mathbb{C}$ be a bounded trilinear map. Let $\widetilde{T}$: $A \to B(V, W^*)$ be the map canonically associated to T i.e.: $\langle (\widetilde{T}(a))(v), w \rangle = T(w, a, v)$.*

Then $\|T\|_{cb} \leq 1$ iff there exist a Hilbert space H, a representation π: $B(E) \to B(H)$ and $\sigma \in CB(V, H_c)$, $\tau \in CB(H_c, W^*)$ such that: $\|\tau\|_{cb} \leq 1$, $\|\sigma\|_{cb} \leq 1$ and: $\forall a \in A$, $\widetilde{T}(a) = \tau \circ \pi(a) \circ \sigma$.

$$\begin{array}{ccc} H & \xrightarrow{\pi(a)} & H \\ \sigma \uparrow & & \downarrow \tau \\ V & \xrightarrow{\widetilde{T}(a)} & W^* \end{array}$$

Proof: We prove the "only if" part. Let us assume that $\|T\|_{cb} \leq 1$. We first apply Theorem 2.1 to T. We then obtain Hilbert spaces H, K_1, K_2, a representation $\pi\colon B(E) \to B(H)$, restrictions of representations $\pi_1\colon V \to B(K_1), \pi_2\colon W \to B(K_2)$ and contractions $T_1\colon \mathbb{C} \to K_1$, $T_2\colon K_2 \to \mathbb{C}$, $S_1\colon K_1 \to H$, $S_2\colon H \to K_2$ such that for all $(y, a, x) \in W \times A \times V$, $T(y, a, x) = T_2\pi_2(y)S_2\pi(a)S_1\pi_1(x)T_1$. We define $\sigma\colon V \to H_c$ and $\tau_1\colon W \to H_r^*$ by $\sigma(x) = S_1\pi_1(x)T_1$ and $\tau_1(y) = T_2\pi_2(y)S_2$. We clearly have $\|\sigma\|_{cb} \leq 1, \|\tau_1\|_{cb} \leq 1$. Moreover, as $\langle\widetilde{T}(a)(x), y\rangle = T(y, a, x)$, we have $\widetilde{T}(a) = \tau_1^* \circ \pi(a) \circ \sigma$. We now let $\tau = \tau_1^*$. Then the result follows from (2.6) and (2.3). The "if" part can be proved by similar arguments; we omit it. ∎

Let V, Z be operator spaces. Following the notation in [ER2], we denote by $\Gamma_2(V, Z)$ the space of linear $\varphi\colon V \to Z$ for which there exist a Hilbert space H, $\sigma \in CB(V, H_c)$, $\tau \in CB(H_c, Z)$ such that $\varphi = \tau \circ \sigma$. We define $\gamma_2(\varphi) = \inf \{\|\tau\|_{cb}\|\sigma\|_{cb}\}$ where the infimum runs over all possible factorizations. γ_2 is a complete norm on $\Gamma_2(V, Z)$. In the particular case $A = \mathbb{C}$, Proposition 2.2 gives us the following identification:

PROPOSITION 2.3 ([ER2, th 5.3]): *Given operator spaces V, W:*

$$\Gamma_2(V, W^*) = (W \overset{h}{\otimes} V)^*.$$

Remark 2.4: In the paper [ER2], Effros and Ruan actually defined an operator space structure on $\Gamma_2(V, W^*)$ for which the identification in Proposition 2.3 holds in the operator space sense. However, we will not make use of this stronger result. ∎

For the computation of the norm in $CB(V, H_c)$ (or in $CB(V, H_r)$) we will use the following simple result which is undoubtly well-known:

PROPOSITION 2.5: *Let V be an operator space, let H be a Hilbert space.*

(i) *For any $u\colon V \to H_c$:*

$$\|u\|_{cb} = \operatorname{Sup} \left\{ \Big(\sum_{i=1}^{n} \|u(x_i)\|^2\Big)^{1/2} \Bigg| \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\| \leq 1 \right\}.$$

(ii) *For any $u\colon V \to H_r$:*

$$\|u\|_{cb} = \operatorname{Sup} \left\{ \Big(\sum_{i=1}^{n} \|u(x_i)\|^2\Big)^{1/2} \Big| \ \|(x_1, \ldots, x_n)\| \leq 1 \right\}.$$

Proof: We only prove (i). We let:

$$|||u||| = \text{Sup}\left\{\Big(\sum_{i=1}^{n}\|u(x_i)\|^2\Big)^{1/2} \Bigm| \left\|\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right\| \leq 1\right\}.$$

For any $h_1, \ldots, h_n$ in H, $\left\|\begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}\right\|_{\mathcal{M}_{n1}(H_c)} = (\sum \|h_i\|^2)^{1/2}$. Therefore, $|||u||| =$ $\text{Sup}\left\{\left\|\begin{pmatrix} u(x_1) \\ \vdots \\ u(x_n) \end{pmatrix}\right\| \Bigm| \left\|\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right\| \leq 1\right\}$, hence $|||u||| \leq \|u\|_{cb}$.

Conversely, let us consider $x = [x_{ij}] \in \mathcal{M}_n(V)$ and $t_1, \ldots, t_n \in \mathbb{C}$.

$$\begin{aligned}\Big(\sum_i\Big\|\sum_j u(x_{ij})t_j\Big\|^2\Big)^{1/2} &= \Big(\sum_i\Big\|u\Big(\sum_j t_jx_{ij}\Big)\Big\|^2\Big)^{1/2} \\ &\leq |||u|||\,\Big\|\Big(\sum_j t_jx_{ij}\Big)_i\Big\|_{\mathcal{M}_{n1}(V)} \\ &\leq |||u|||\Big(\sum_j |t_j|^2\Big)^{1/2}\|x\|.\end{aligned}$$

This proves that $\|u\|_{cb} \leq |||u|||$. ∎

3. The main result

Our main result is an abstract factorization theorem from which Theorem 1.1 will be deduced in section 4. It may be seen as an analytic version of Proposition 2.2 in the case when A is a C*-algebra.

THEOREM 3.1: *Let X, Y be operator spaces. Let A be a C^*-algebra.*

Let $\varphi \in H^\infty((Y \overset{h}{\otimes} A \overset{h}{\otimes} X)^*)$. Then there exist a Hilbert space H, a representation π: $A \to B(H)$, $\varphi_1 \in H^\infty(CB(X, H_c))$ and $\varphi_2 \in H^\infty(CB(Y, H_r^*))$ such that:

$$\|\varphi_1\|_\infty\ \|\varphi_2\|_\infty \leq \|\varphi\|_\infty \quad \textit{and}$$

$$\forall (z, y, a, x) \in D \times Y \times A \times X\colon\quad \varphi(z)(y, a, x) = \langle \pi(a)\varphi_1(z)(x), \varphi_2(z)(y)\rangle.$$

The proof of Theorem 3.1 will be based on three lemmas. The first one deals with the classical Hardy spaces of vector valued analytic functions. Let us recall their usual definition.

Let X be a Banach space and let $p \in [1, +\infty[$. We let $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and we denote by dt the normalized Lebesgue measure on $\mathbb{T}$. For an analytic function $x: D \to X$, we let

$$\|x\|_p = \operatorname*{Sup}_{0<r<1} \left(\int_{\mathbb{T}} \|x(re^{it})\|_X^p dt \right)^{1/p}.$$

We denote by $H^p(X)$ the space of those functions for which $\|x\|_p < +\infty$. As is well-known, $(H^p(X), \| \ \|_p)$ is a Banach space. This definition clearly generalizes that of $H^\infty(X)$ given in the introduction of this paper. We simply denote by H^p the space $H^p(\mathbb{C})$. Let us denote by $S_X \in B(H^p(X))$ the usual shift operator on $H^p(X)$ defined by:

$$\forall x \in H^p(X), \forall z \in D, \quad (S_X(x))(z) = zx(z).$$

Definition 3.2: Let E, F be Hilbert spaces. Given $u \in B(H^2(E), H^2(F))$, we say that u is a module map provided that $u \circ S_E = S_F \circ u$. We denote by $mB(H^2(E), H^2(F))$ the subspace of these maps. Let G be an operator space. Given $u \in CB(G, B(H^2(E), H^2(F)))$, we say that u is a module c.b. map provided that $u(g)$ is a module map for any $g \in G$. We denote by $mCB(G, B(H^2(E), H^2(F)))$ the subspace of these maps. ∎

We have an obvious identification:

$$mCB(G, B(H^2(E), H^2(F))) = CB(G, mB(H^2(E), H^2(F))). \tag{3.1}$$

The following result is well-known; we omit its proof.

SUBLEMMA 3.3: *Let E, F be Hilbert spaces. Let $\varphi \in H^\infty(B(E,F))$. We define $\widehat{\varphi}$: $H^2(E) \to H^2(F)$ by $\widehat{\varphi}(e)(z) = \varphi(z)(e(z))$ for any $e \in H^2(E)$. Then the linear map $\varphi \mapsto \widehat{\varphi}$ is an isometric isomorphism from $H^\infty(B(E,F))$ onto $mB(H^2(E), H^2(F))$.*

We wish to settle an analogous identification result in the framework of c.b. maps.

LEMMA 3.4: *Let G be an operator space. Let E, F be Hilbert spaces. For any $\varphi \in H^\infty(CB(G, B(E,F)))$, we define $\tau(\varphi)$: $G \to mB(H^2(E), H^2(F))$ by:*

$$\tau(\varphi)(g)(e)(z) = (\varphi(z)(g))(e(z)) \ \text{ for any } g \in G, e \in H^2(E).$$

Then τ: $H^\infty(CB(G,B(E,F))) \to mCB(G,B(H^2(E),H^2(F)))$ *is an isometric isomorphism.*

Proof: Sublemma 3.3 yields a natural inclusion

$$H^\infty(B(E,F)) \subset B(H^2(E),H^2(F)).$$

This embedding defines an operator space structure on $H^\infty(B(E,F))$. It is worthwhile to notice that we have an isometric isomorphism:

$$\mathcal{M}_n(H^\infty(B(E,F))) = H^\infty(\mathcal{M}_n(B(E,F))). \tag{3.2}$$

That folllows from Sublemma 3.3 again.

Now let $\varphi \in H^\infty(CB(G,B(E,F)))$. We may define $\overline{\varphi}$: $G \to H^\infty(B(E,F))$ by $\overline{\varphi}(g)(z) = \varphi(z)(g)$ for any $g \in G$. Since

$$\|\varphi\|_\infty = \text{ Sup } \{\|\varphi(z)^{(n)}(g)\| \ / \ z \in D, n \geq 1, g \in \mathcal{M}_n(G), \|g\| \leq 1\}$$

it follows from (3.2) that $\|\varphi\|_\infty = \|\overline{\varphi}\|_{cb}$. Consequently, we have

$$H^\infty(CB(G,B(E,F))) = CB(G,mB(H^2(E),H^2(F))).$$

From (3.1) we then get:

$$H^\infty(CB(G,B(E,F))) = mCB(G,B(H^2(E),H^2(F))).$$

It is now easy to check that the isometric isomorphism which provides the identification above is τ, and the proof is complete. ∎

We now turn to the second lemma. It is the most important tool in the proof of Theorem 3.1. It should be compared with Proposition 2.2. Note that in this latter result, the space A was only assumed to be an operator space. See Remark 3.8 for more about this.

LEMMA 3.5: *Let* V,W *be two operator spaces. Let* A *be a* C^**-algebra. Let* T: $W \times A \times V \to \mathbb{C}$ *be a completely contractive trilinear map. Let* $\widetilde{T}$: $A \to B(V,W^*)$ *be the map canonically associated to* T. *Assume that we are given two c.c. maps* s_1: $V \to V$ *and* s_2: $W \to W$ *such that:* $\forall a \in A$, $\widetilde{T}(a) = s_2^*\widetilde{T}(a)s_1$.

Then there exist a Hilbert space K, *a representation* ρ: $A \to B(K)$, *an isometry* $S \in B(K)$ *and* $\alpha \in CB(V,K_c)$, $\beta \in CB(K_c,W^*)$ *such that:*

$$\text{(i)} \quad \|\alpha\|_{cb} \leq 1, \quad \text{(ii)} \quad \|\beta\|_{cb} \leq 1, \tag{3.3}$$

$$\alpha s_1 = S\alpha, \tag{3.4}$$

$$\forall a \in A, \quad \rho(a)S = S\rho(a), \tag{3.5}$$

$$\forall a \in A, \quad \widetilde{T}(a) = \beta\rho(a)\alpha. \tag{3.6}$$

Remark 3.6: (i) In the statement above, we may always assume that $K = \overline{\mathrm{Span}}\{\rho(A)\alpha(V)\}$ by using obvious restrictions. Hence we get an additional property for free:

$$s_2^*\beta S = \beta. \tag{3.7}$$

Indeed, given $a \in A$, we have:

$$\begin{aligned} s_2^*\beta S\rho(a)\alpha &= s_2^*\beta\rho(a)\alpha s_1 \quad \text{by (3.5) and (3.4)} \\ &= s_2^*\widetilde{T}(a)s_1 \quad \text{by (3.6)} \\ &= \widetilde{T}(a) \\ &= \beta\rho(a)\alpha \quad \text{by (3.6) again.} \end{aligned}$$

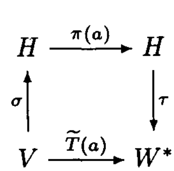
Since $K = \overline{\mathrm{Span}}\{\rho(A)\alpha(V)\}$, we obtain (3.7). ∎

(ii) Let us assume that there is a map r: $W^* \to W^*$ such that $s_2^*r = \mathrm{Id}_{W^*}$ and $\widetilde{T}(a)s_1 = r\widetilde{T}(a)$ for any $a \in A$. Of course, we have $\widetilde{T}(a) = s_2^*\, \widetilde{T}(a)s_1$ for any $a \in A$. The interesting point in this particular case is that (3.7) is strengthened as follows:

$$\beta S = r\beta. \tag{3.8}$$

∎

Proof of Lemma 3.5: We may apply Proposition 2.2 to the trilinear map T. Hence we get the following factorization diagram:



for some Hilbert space H, a representation $\pi\colon A \to B(H)$ and completely contractive maps $\sigma \in CB(V, H_c), \tau \in CB(H_c, W^*)$. For any $N \geq 1$, we define H_N as the direct sum of N copies of H with the following norm:

$$\forall (h_1, \ldots, h_N) \in H \times \cdots \times H, \; \|(h_1, \ldots, h_N)\| = \left(\frac{1}{N} \sum_{k=1}^{N} \|h_k\|^2 \right)^{1/2}.$$

Let $\mathcal{F}$ be a free ultrafilter on $\mathbb{N}^*$ and let $\widehat{H} = \left(\prod_{N\geq 1} H_N\right)/\mathcal{F}$ be the ultraproduct of $(H_N)_{N\geq 1}$ corresponding to $\mathcal{F}$.

Let $N \geq 1$. We define $\alpha_N\colon V \to H_N$ by $\alpha_N(x) = (\sigma(x), \sigma s_1(x), \sigma s_1^2(x), \ldots, \sigma s_1^{N-1}(x))$. Clearly, we have $\|\alpha_N\| \leq \|\sigma\|$. Therefore we can define a bounded linear map $\alpha_0\colon V \to \widehat{H}$ by letting $\alpha_0(x) = (\alpha_N(x))_{N\geq 1}$.

The main point in this definition is that for any $x \in V$:

$$\|\alpha_0(x)\| = \lim_{N\in\mathcal{F}} \left(\frac{1}{N} \sum_{k=0}^{N-1} \|\sigma s_1^k(x)\|^2 \right)^{1/2} \tag{3.9}$$

We claim that:

$$\alpha_0 \in CB(V, \widehat{H}_c), \quad \|\alpha_0\|_{cb} \leq 1. \tag{3.10}$$

To check this, we use Proposition 2.5. Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathcal{M}_{n1}(V)$. For any $k \geq 0$, we have:

$$\sum_{i=1}^{n} \left\|\sigma\, s_1^k(x_i)\right\|^2 \leq \left\|\sigma\, s_1^k\right\|_{cb}^2 \|x\|^2.$$

Since $\|\sigma\|_{cb} \leq 1$ and $\|s_1\|_{cb} \leq 1$, we have $\left\|\sigma\, s_1^k\right\|_{cb} \leq 1$. Therefore, we obtain for any $N \geq 1$:

$$\sum_{i=1}^{n} \frac{1}{N} \sum_{k=0}^{N-1} \left\|\sigma\, s_1^k(x_i)\right\|^2 \leq \|x\|^2. \tag{3.11}$$

Thus, (3.9) and (3.11) imply: $\sum_{i=1}^{n} \|\alpha_0(x_i)\|^2 \leq \|x\|^2$. The statement (3.10) clearly follows from this inequality and Proposition 2.5 (i).

Let $N \geq 1$ and $a \in A$. We denote by $\pi_N(a)$ the operator on H_N defined by the action of $\pi(a)$ on each component of $H_N = H \oplus \cdots \oplus H$. Obviously, $\pi_N\colon A \to$

$B(H_N)$ is a representation, hence we can define a representation $\hat{\pi}$: $A \to B(\widehat{H})$ by letting:

$$\hat{\pi}(a)\big((\lambda_N)_{N\geq 1}\big) = \big(\pi_N(a)\lambda_N\big)_{N\geq 1} \tag{3.12}$$

for any $a \in A$ and $\lambda_N \in H_N$.

Now let $K = \overline{\text{Span}\{\hat{\pi}(A)\alpha_0(V)\}}$. Since A is an algebra, K is an invariant subspace of $\widehat{H}$ for any $\hat{\pi}(a)$. Therefore we can define a representation ρ: $A \to B(K)$ by restricting $\hat{\pi}$ to K. Denoting by P_K the orthogonal projection from $\widehat{H}$ onto K, we let $\alpha = P_K\alpha_0$. Of course (3.10) leads to (3.3)(i). Note that we have $\rho(a)\alpha = \hat{\pi}(a)\alpha_0$ for any $a \in A$. The interesting point in the definitions of α and ρ is that:

$$\forall a_i \in A, \forall x_i \in V, \quad \Big\|\sum_{i=1}^{p} \rho(a_i)\alpha(x_i)\Big\| = \Big\|\sum_{i=1}^{p} \rho(a_i)\alpha s_1(x_i)\Big\|. \tag{3.13}$$

Indeed, for any $N \geq 1$, we have:

$$\begin{aligned} &\Big\|\sum_{i=1}^{p} \pi_N(a_i)\alpha_N(x_i)\Big\|^2 - \Big\|\sum_{i=1}^{p} \pi_N(a_i)\alpha_N s_1(x_i)\Big\|^2 \\ &= \frac{1}{N}\left(\Big\|\sum_{i=1}^{p} \pi(a_i)\sigma\, x_i\Big\|^2 - \Big\|\sum_{i=1}^{p} \pi(a_i)\sigma\, s_1^N(x_i)\Big\|^2\right). \end{aligned} \tag{3.14}$$

Clearly, the $\mathcal{F}$-limit of the right member in (3.14) is zero. On the other hand, the left member tends to $\|\sum_{i=1}^p \rho(a_i)\alpha(x_i)\|^2 - \|\sum_{i=1}^p \rho(a_i)\alpha s_1(x_i)\|^2$ by (3.12), and then (3.13) follows. This property (3.13) allows us to define an isometry $S \in B(K)$ by letting: $S\big(\sum_{i=1}^p \rho(a_i)\alpha(x_i)\big) = \sum_{i=1}^p \rho(a_i)\alpha s_1(x_i)$. We clearly obtain:

$$\forall a \in A, \quad S\,\rho(a)\alpha = \rho(a)\,\alpha s_1. \tag{3.15}$$

Let $a, b \in A$.

$$\begin{aligned} S\,\rho(a)\rho(b)\alpha &= S\,\rho(ab)\alpha \\ &= \rho(ab)\alpha s_1 \quad \text{by (3.15)} \\ &= \rho(a)\,S\,\rho(b)\,\alpha \quad \text{by (3.15) again.} \end{aligned}$$

Therefore, S and ρ satisfy (3.5).

Let us assume that A is unital. Then $\rho(1)$ is the identity map Id_K. Therefore (3.4) follows from (3.15). In the case when A has merely an approximate identity (e_t), the net $\rho(e_t)$ converges strongly to Id_K and we may argue as above to get (3.4).

To complete the proof of this lemma, it remains to define a completely contractive map $\beta \in CB(K_c, W^*)$ which satisfies (3.6). For this purpose, the crucial step will be the following:

CLAIM: *$\forall y = (y_1, \ldots, y_n) \in \mathcal{M}_{1n}(W)$, $\forall a_{qj} \in A$, $\forall x_{qj} \in V$ $(1 \leq j \leq n,\ 1 \leq q \leq p)$:*

$$\Big|\sum_{j=1}^{n}\langle y_j, \sum_{q=1}^{p} \widetilde{T}(a_{qj})(x_{qj})\rangle\Big| \leq \|y\|\Big(\sum_{j=1}^{n}\Big\|\sum_{q=1}^{p}\rho(a_{qj})\alpha(x_{qj})\Big\|^2\Big)^{1/2}.$$

Proof of the claim: Given $k \geq 0$, we have

$$\begin{aligned}
&|\sum_j\langle y_j, \sum_q \widetilde{T}(a_{qj})(x_{qj})\rangle|^2\\
&= |\sum_j\langle y_j, \sum_q s_2^{*k}\widetilde{T}(a_{qj})s_1^k(x_{qj})\rangle|^2\\
&= |\sum_j\langle s_2^k(y_j), \sum_q \widetilde{T}(a_{qj})s_1^k(x_{qj})\rangle|^2\\
&= |\sum_j\langle s_2^k(y_j), \sum_q \tau\pi(a_{qj})\sigma\, s_1^k(x_{qj})\rangle|^2\\
&\leq \|\tau^* s_2^k\|_{cb}^2\,\|y\|^2\sum_j\|\sum_q \pi(a_{qj})\sigma\, s_1^k(x_{qj})\|^2 \quad \text{by (2.6).}
\end{aligned}$$

Since $\|\tau\|_{cb} \leq 1$ and $\|s_2\|_{cb} \leq 1$, we have $\|\tau^* s_2^k\|_{cb} \leq 1$ by (2.3). Therefore, we obtain for any $N \geq 1$:

$$\Big|\sum_j\langle y_j, \sum_q \widetilde{T}(a_{qj})(x_{qj})\rangle\Big|^2 \leq \|y\|^2\sum_j\Big(\frac{1}{N}\sum_{k=0}^{N-1}\|\sum_q \pi(a_{qj})\,\sigma\, s_1^k(x_{qj})\|^2\Big)$$

Passing to the $\mathcal{F}$-limit, we obtain the announced result. ∎

We are now ready to end the proof of Lemma 3.5. In the particular case when $n = 1$, the claim yields:

$$|\langle y, \sum_q \widetilde{T}(a_q)(x_q)\rangle| \leq \|y\|\ \|\sum_q \rho(a_q)\alpha(x_q)\| \tag{3.16}$$

for any $y \in W, a_q \in A, x_q \in V$.

That means that we may define a contractive map $\beta\colon K \to W^*$ by letting

$$\beta\Big(\sum_q \rho(a_q)\alpha(x_q)\Big) = \sum_q \widetilde{T}(a_q)(x_q).$$

Of course, β satisfies (3.6). Hence the remainder point is the fact that β is c.c. from K_c into W^*. We let $y = (y_1, \ldots, y_n) \in \mathcal{M}_n(W)$ and $k_1, \ldots, k_n \in \mathrm{Span}\{\rho(A)\alpha(V)\}$. Each vector k_j may be written $k_j = \sum_q \rho(a_{qj})\alpha(x_{qj})$ for some $x_{qj} \in V, a_{qj} \in A$. Note that $\beta(k_j) = \sum_q \widetilde{T}(a_{qj})(x_{qj})$. Therefore our claim means that:

$$\Big|\sum_j \langle y_j, \beta(k_j)\rangle\Big| \leq \|y\| \, \Big(\sum_j \|k_j\|^2\Big)^{1/2}. \tag{3.17}$$

We deduce from (3.17) and Proposition 2.5 (ii) that $\beta^*_{/W}\colon W \to K_r^*$ is a c.c. map and then, (3.3)(ii) follows by (2.3). ∎

Lemma 3.7: *Let H be a Hilbert space. Let A be a C^*-algebra. Let $\rho\colon A \to B(H^2(H))$ be a representation such that $\rho(a)S_H = S_H\rho(a)$ for any $a \in A$. Then there exists a representation $\pi\colon A \to B(H)$ such that:*

$$\forall\, a \in A, \quad \pi(a) \otimes \mathrm{Id}_{H^2} = \rho(a).$$

Proof: We use the well-known identification $H = H^2(H) \ominus S_H(H^2(H))$;

$$\forall (h, f, a) \in H \times H^2(H) \times A, \quad \langle \rho(a)h, S_H(f)\rangle = \langle h, S_H\rho(a^*)f\rangle = 0.$$

Therefore $\rho(a)(H) \subset H$ for any $a \in A$. Let us denote by $\pi(a) \in B(H)$ the restriction of $\rho(a)$ to H. The resulting representation $\pi\colon A \to B(H)$ is convenient. ∎

Proof of Theorem 3.1: Let $\varphi \in H^\infty\big((Y \overset{h}{\otimes} A \overset{h}{\otimes} X)^*\big)$ with $\|\varphi\|_\infty = 1$. Following Lemma 3.4, we define $\tau(\varphi) \in CB\big((Y \overset{h}{\otimes} A \overset{h}{\otimes} X), B(H^2)\big)$ as the module c.b. map associated to φ. Letting $V = X \overset{h}{\otimes} (H^2)_c$ and $W = (H^2)_r^* \overset{h}{\otimes} Y$, we know from (2.4) that

$$CB\big((Y \overset{h}{\otimes} A \overset{h}{\otimes} X), B(H^2)\big) \cong \big(W \overset{h}{\otimes} A \overset{h}{\otimes} V\big)^*.$$

We denote by $T\colon W \times A \times V \to \mathbb{C}$ the trilinear c.c. map corresponding to $\tau(\varphi)$ under this identification. We wish to apply Lemma 3.5 to T. First recall that $S_{\mathbb{C}}$

denotes the shift operator on H^2. Analogously, we denote by $\overline{S_{\mathbb{C}}}$ the corresponding isometry on $(H^2)^*$. We now let $s_1 = \mathrm{Id}_X \otimes S_{\mathbb{C}}\colon V \to V$ and $s_2 = \overline{S_{\mathbb{C}}} \otimes \mathrm{Id}_Y : W \to W$. It is easy to check that s_1 and s_2 are c.c. maps (see [BP prop. 5.11] for example). Secondly, using (2.5), we obtain $W^* \cong CB(Y, (H^2)_c)$. Let $r\colon W^* \to W^*$ be the map induced by $S_{\mathbb{C}}$. Namely, we let $(r(u))(y) = S_{\mathbb{C}}(u(y))$ for any $u \in CB(Y, (H^2)_c)$ and $y \in Y$. We clearly have:

$$s_2^* r = \mathrm{Id}_{W^*}. \tag{3.18}$$

Now let $a \in A$, $x \in X$, $f \in H^2$, $y \in Y$, $z \in D$. The formula

$$(\widetilde{T}(a)(x \otimes f))(y)(z) = \varphi(z)(y, a, x).\, f(z)$$

shows that:

$$\forall a \in A, \quad \widetilde{T}(a) s_1 = r \widetilde{T}(a). \tag{3.19}$$

The above properties (3.18) and (3.19) allow us to apply Lemma 3.5 and Remark 3.6 (ii). This yields a Hilbert space K, an isometry $S \in B(K)$, a representation $\rho\colon A \to B(K)$ and two c.c. maps $\alpha \in CB(V, K_c)$, $\beta \in CB(K_c, W^*)$ which satisfy (3.4), (3.5), (3.6) and (3.8).

Let $K_0 = \bigcap_{n\geq 0} S^n(K)$ and $H = K \ominus S(K)$. The space K_0 is reducing for S and Wold's decomposition lemma states that $K = K_0 \overset{\perp}{\oplus} H^2(H)$ with $S_{/H^2(H)} = S_H$. We claim that:

$$\beta_{/K_0} = 0. \tag{3.20}$$

To check this, we fix $k \in K_0$ and $y \in Y$. For any $n \geq 0$, there exists $k_n \in K$ such that $k = S^n(k_n)$. From (3.8), we get $\beta(k) = r^n \beta(k_n)$ hence $(\beta(k))(y) \in S_{\mathbb{C}}^n(H^2)$. Since $\bigcap_{n\geq 0} S_{\mathbb{C}}^n(H^2) = \{0\}$, the statement (3.20) follows.

Moreover, (3.5) ensures that $\rho(a)(K_0) \subset K_0$ for any $a \in A$. Consequently, we have:

$$\forall\, a \in A, \quad \rho(a)(H^2(H)) \subset H^2(H). \tag{3.21}$$

Let us denote by q the orthogonal projection from K onto $H^2(H)$. Replacing α by $q\alpha, \rho(a)$ by $\rho(a)_{/H^2(H)}$ and β by $\beta_{/H^2(H)}$, we get from (3.20) and (3.21) that we can assume that:

$$K = H^2(H)\ ,\ S = S_H. \tag{3.22}$$

This reduction is very interesting since it enables us to identify α and β with analytic functions valued in adapted spaces of c.b. maps. First note that $\alpha \in CB(V, (H^2(H))_c)$. Now:

$$\begin{aligned} CB(V, (H^2(H))_c) &= \big((H^2(H))_r^* \overset{h}{\otimes} V\big)^* \quad \text{by (2.5)} \\ &= \big((H^2(H))_r \overset{h}{\otimes} X \overset{h}{\otimes} (H^2)_c\big)^* \\ &= CB(X, B(H^2, H^2(H))) \quad \text{by (2.4)}. \end{aligned}$$

The map $\hat{\alpha} \in CB(X, B(H^2, H^2(H)))$ corresponding to α under the above identification is defined by: $\forall x \in X$, $\forall f \in H^2$, $\hat{\alpha}(x)(f) = \alpha(x \otimes f)$. Taking into account (3.22), the commutation property (3.4) exactly means that $\hat{\alpha}$ is a module c.b. map. By Lemma 3.4 we deduce that there exist $\varphi_1 \in H^\infty(CB(X, H_c))$ with $\|\varphi_1\|_\infty \le 1$ and such that:

$$(3.23) \qquad \forall x \in X, \forall z \in D, \quad (\varphi_1(z))(x) = (\alpha(x \otimes 1))(z).$$

A similar study of $\beta \in CB((H^2(H))_c, W^*)$ leads to $\varphi_2 \in H^\infty(CB(Y, H_r^*))$ with $\|\varphi_2\|_\infty \le 1$ and such that:

$$(3.24) \qquad \forall y \in Y,\ \forall h \in H, \forall\ z \in D, \quad \langle(\varphi_2(z))(y), h\rangle = (\beta(h \otimes 1)(y))(z),$$

From (3.5), (3.22) and Lemma 3.7, we know that there is a representation $\pi\colon A \to B(H)$ such that:

$$(3.25) \qquad \forall a \in A,\ \pi(a) \otimes \mathrm{Id}_{H^2} = \rho(a).$$

Let $a \in A, x \in X, y \in Y$. We define $f \in H^2(H)$ by letting $f(z) = \pi(a)(\varphi_1(z)(x))$. It follows from (3.23) and (3.25) that $f = \rho(a)\alpha(x \otimes 1)$. Reminding the reader that $(\widetilde{T}(a)(x \otimes 1))(y) = \varphi(.)(y, a, x)$, we obtain from (3.6) that $(\beta(f))(y) = \varphi(.)(y, a, x)$. On the other hand, (3.24) implies that

$$\forall z \in D, \quad (\beta(f)(y))(z) = \langle \varphi_2(z)(y), f(z)\rangle.$$

Hence we have

$$\forall z \in D, \quad \langle \pi(a)\varphi_1(z)(x), \varphi_2(z)(y)\rangle = \varphi(z)(y, a, x)$$

and this completes the proof. ∎

Remark 3.8: It is easy to check that Lemma 3.5 can be stated under the assumption that A is a unital operator algebra. In this case, we obtain a c.c. homomorphism $\rho: A \to B(K)$ instead of a representation. However, in the proof of Theorem 3.1, the self-adjointness of A is used in order to prove Lemma 3.7 and the property (3.21).

4. The analytic form of Christensen and Sinclair's theorem

Our main goal is now to prove Theorem 1.1. As will be seen below, its proof is a formal combination of two results which are both straightforward corollaries of Theorem 3.1. These useful results are obtained by applying Theorem 3.1 in two particular cases which are successively:

(a) $X = E_c, Y = F_r^*$ when E, F are Hilbert spaces.

(b) $A = \mathbb{C}$.

In case (a), $(Y \overset{h}{\otimes} A \overset{h}{\otimes} X)^* = (F_r^* \overset{h}{\otimes} A \overset{h}{\otimes} E_c)^* = CB(A, B(E,F))$ by (2.4). Thus Theorem 3.1 is a factorization result for any $\varphi \in H^\infty(CB(A, B(E,F))$. It is not hard to check that this result is nothing but Theorem 1.1 in the particular case $n = 1$.

In case (b), $(Y \overset{h}{\otimes} A \overset{h}{\otimes} X)^* = (Y \overset{h}{\otimes} X)^* = \Gamma_2(X, Y^*)$ by Proposition 2.3. Then Theorem 3.1 provides a factorization result which roughly says that for any $\varphi \in H^\infty(\Gamma_2(X, Y^*))$, the column Hilbert space factorization of $\varphi(z)$ may be achieved in a way that preserves analyticity.

Let us now formulate these two results more precisely:

PROPOSITION 4.1: *Let A be a C^*-algebra. Let E, F be Hilbert spaces. Let $\varphi \in H^\infty(CB(A, B(E,F)))$. Then there exist a Hilbert space H, a representation $\pi: A \to B(H), \varphi_1 \in H^\infty(B(E,H))$ and $\varphi_2 \in H^\infty(B(H,F))$ such that $\|\varphi_1\|_\infty \|\varphi_2\|_\infty \leq \|\varphi\|_\infty$ and:*

$$\forall z \in D, \forall a \in A, \quad \varphi(z)(a) = \varphi_2(z)\pi(a)\varphi_1(z).$$

PROPOSITION 4.2: *Let X, Y be operator spaces. Let $\varphi \in H^\infty(\Gamma_2(X, Y^*))$. Then there exist a Hilbert space H, $\varphi_1 \in H^\infty(CB(X, H_c))$, and $\varphi_2 \in H^\infty(CB(Y, H_r^*))$ such that $\|\varphi_1\|_\infty \|\varphi_2\|_\infty \leq \|\varphi\|_\infty$ and:*

$$\forall z \in D, \quad \varphi(z) = (\varphi_2(z))^* \circ \varphi_1(z).$$

Remark 4.3: In Proposition 4.1, we could have written $\varphi_1 \in H^\infty(CB(E_c, H_c))$, $\varphi_2 \in H^\infty(CB(H_c, F_c))$. However, it is not really a stronger result since we have $B(E,H) = CB(E_c, H_c)$ (and $B(H,F) = CB(H_c, F_c)$). This well-known result is an easy consequence of (2.4).

Proof of Theorem 1.1: In view of Proposition 4.1, we may assume that $n \geq 2$. We let $\varphi \in H^\infty(CB(A_n \times \cdots \times A_1, B(E,F))$. We may assume that $\|\varphi\|_\infty = 1$. Taking into account the definition of c.b. multilinear maps and (2.4), we have $CB(A_n \times \cdots \times A_1, B(E,F)) = \big(F_r^* \overset{h}{\otimes} A_n \overset{h}{\otimes} \cdots \overset{h}{\otimes} A_1 \overset{h}{\otimes} E_c\big)^*$. Letting $Y = F_r^* \overset{h}{\otimes} A_n$ and $X = A_{n-1} \overset{h}{\otimes} \cdots \overset{h}{\otimes} A_1 \overset{h}{\otimes} E_c$, we thus may consider φ as an element of $H^\infty(\Gamma_2(X, Y^*))$ with corresponding norm equal to one. Let us apply Proposition 4.2 to φ. This yields a Hilbert space H_{n-1} and analytic functions

$$\widetilde{\theta}_n \in H^\infty(CB(A_{n-1} \overset{h}{\otimes} \cdots \overset{h}{\otimes} A_1 \overset{h}{\otimes} E_c, (H_{n-1})_c)$$

and

$$\widetilde{\psi}_n \in H^\infty(CB(F_r^* \overset{h}{\otimes} A_n, (H_{n-1})_r^*))$$

such that $\|\widetilde{\psi}_n\|_\infty \leq 1, \|\widetilde{\theta}_n\|_\infty \leq 1$ and:

$$\begin{aligned} &\forall a_n \times \cdots \times a_1 \in A_n \times \cdots \times A_1, \forall (e, f^*) \in E \times F^*, \forall z \in D \\ &\langle \varphi(z)(a_n, \ldots, a_1)e, f^* \rangle = \langle \widetilde{\psi}_n(z)(f^* \otimes a_n), \widetilde{\theta}_n(z)\ (a_{n-1} \otimes \cdots \otimes a_1 \otimes e) \rangle. \end{aligned} \tag{4.1}$$

Applying (2.4) twice, one gets:

$$CB(F_r^* \overset{h}{\otimes} A_n, (H_{n-1})_r^*) = CB(A_n, B(H_{n-1}, F)). \tag{4.2}$$

Similarly,

$$CB\big(A_{n-1} \overset{h}{\otimes} \cdots \overset{h}{\otimes} A_1 \overset{h}{\otimes} E_c, (H_{n-1})_c\big) = CB\big(A_{n-1} \overset{h}{\otimes} \cdots \overset{h}{\otimes} A_1, B(E, H_{n-1})\big),$$

hence we have:

$$CB\big(A_{n-1} \overset{h}{\otimes} \cdots \overset{h}{\otimes} A_1 \overset{h}{\otimes} E_c, (H_{n-1})_c\big) = CB\big(A_{n-1} \times \cdots \times A_1, B(E, H_{n-1})\big). \tag{4.3}$$

Let us denote by

$$\psi_n \in H^\infty(CB(A_n, B(H_{n-1}, F))),\ \theta_n \in H^\infty(CB(A_{n-1} \times \cdots \times A_1, B(E, H_{n-1})))$$

the analytic functions corresponding to $\widetilde{\psi}_n$ and $\widetilde{\theta}_n$ under the identifications (4.2) and (4.3). In this setting, (4.1) becomes:

$$\text{(4.4)}\qquad \begin{gathered} \forall (a_n,\ldots,a_1)\in A_n\times\cdots\times A_1,\ \forall z\in D:\\ \varphi(z)(a_n,\ldots,a_1)=\psi_n(z)(a_n)\circ\theta_n(z)(a_{n-1},\ldots,a_1). \end{gathered}$$

It is clear that this process can be iterated. We thus obtain by induction that there exist Hilbert spaces $H_1,\ldots,H_{n-1}$ and analytic functions

$$\psi_1\in H^\infty(CB(A_1,B(E,H_1))),\quad \psi_n\in H^\infty(CB(A_n,B(H_{n-1},F)))$$
$$\text{and}\quad \psi_i\in H^\infty(CB(A_i,B(H_{i-1},H_i)))\quad (2\le i\le n-1)$$

such that $\|\psi_i\|_\infty\le 1$ for all $1\le i\le n$ and:

$$\text{(4.5)}\qquad \begin{gathered} \forall (a_n,\ldots,a_1)\in A_n\times\cdots\times A_1,\ \forall z\in D:\\ \varphi(z)(a_n,\ldots,a_1)=(\psi_n(z))(a_n)\circ(\psi_{n-1}(z))(a_{n-1})\circ\cdots\circ(\psi_1(z))(a_1). \end{gathered}$$

We now apply Proposition 4.1 to each bounded analytic function ψ_i and the result follows. ∎

Remark 4.4: It should be noticed that in the decomposition result stated as (4.5), we do not need $A_1,\ldots,A_n$ to be C^*-algebras. This result is valid even when $A_1,\ldots,A_n$ are merely operator spaces.

Remark 4.5: Actually Proposition 4.2 is true without assuming the second space to be a dual space. Namely we have:

Let X and Z be operator spaces. Let $\varphi\in H^\infty(\Gamma_2(X,Z))$. Then there exist a Hilbert space H, $\varphi_1\in H^\infty(CB(X,H_c))$, $\psi_1\in H^\infty(CB(H_c,Z))$ such that $\|\varphi_1\|_\infty\|\psi_1\|_\infty\le\|\varphi\|_\infty$ and $\forall z\in D,\quad \varphi(z)=\psi_1(z)\circ\varphi_1(z)$.

Indeed, since the natural inclusion $Z\subset Z^{**}$ is completely isometric (by (2.3)), we can consider φ as an element of $H^\infty(\Gamma_2(X,Z^{**}))$. Thus we can apply to φ the reasoning which led us to Theorem 3.1 in the particular case $Y=Z^*,\ A=\mathbb{C}$. Keeping the notation used in the proof of this theorem, we just have to check that for any $z\in D$, $\varphi_2(z)^*$ is valued in Z. Let us denote by $\mathcal{V}\subset CB(Y,(H^2)_c)$ the space of c.b. maps $v\colon Y\to(H^2)_c$ such that for any $z\in D$, $y\mapsto(v(y))(z)$ belongs to Z. $\mathcal{V}$ is nothing but the space of weak-star continuous c.b. maps from $Y=Z^*$ into $(H^2)_c$. It is not hard to check that the map $\beta\colon H^2(H)\to CB(Y,(H^2)_c)$ which is built using Lemma 3.5 is valued in $\mathcal{V}$. Therefore the identity (3.24) implies that for any $z\in D$, $\varphi_2(z)^*$ is valued in Z. ∎

Remark 4.6: Let $p \in [1, +\infty]$. A well-known fact about vector valued H^p-spaces is that for any Banach space Z, any $\varphi \in H^p(Z)$ can be written as $\varphi = f\psi$ with $f \in H^p, \psi \in H^\infty(Z)$ and $\|f\|_p \|\psi\|_\infty = \|\varphi\|_p$. Therefore we may easily deduce H^p-versions of Theorem 1.1 from the previous H^∞-version. In the following we keep the notation of Theorem 1.1. Let $1 \leq r_1, \dots, r_n \leq +\infty$ such that $1/p = \sum_{i=1}^n 1/r_i$.

Consider $\varphi \in H^p(CB(A_n \times \cdots \times A_1, B(E,F)))$. Then there exist spaces K_i, representations $\pi_i (1 \leq i \leq n)$ as in Theorem 1.1, $\varphi_0 \in H^{r_0}(B(E, K_1))$, $\varphi_n \in H^{r_n}(B(K_n, F))$ and $\varphi_i \in H^{r_i}(B(K_i, K_{i+1}))(1 \leq i \leq n-1)$ such that:

(i) $\forall (a_n, \cdots, a_1) \in A_n \times \cdots \times A_1$, $\forall z \in D$:

$$\varphi(z)(a_n, \dots, a_1) = \varphi_n(z)\pi_n(a_n)\varphi_{n-1}(z)\dots\varphi_1(z)\pi_1(a)\varphi_0(z).$$

(ii) $\|\varphi_0\|_{r_0} \ \|\varphi_1\|_{r_1} \cdots \|\varphi_n\|_{r_n} \leq \|\varphi\|_p$.

Of course, such a generalization to H^p-spaces is also practicable in Theorem 3.1. We omit the obvious statement. ∎

Remark 4.7: As a consequence of Proposition 4.2, we recover Sarason's factorization theorem which can be formulated as follows [S]:

> Let E, F be Hilbert spaces. For any $\psi \in H^1(C_1(E,F))$, there exist a Hilbert space H and two functions $\psi_1 \in H^2(C_2(E,H))$, $\psi_2 \in H^2(C_2(H,F))$ such that $\|\psi_1\|_2 \|\psi_2\|_2 \leq \|\psi\|_1$ and: $\forall z \in D, \psi(z) = \psi_2(z) \circ \psi_1(z)$.

Here C_1 denotes the space of all nuclear operators whereas C_2 denotes the space of all Hilbert–Schmidt operators. Let us sketch the proof of Sarason's theorem. First note that $CB(E_r, H_c) = C_2(E, H)$ for any Hilbert spaces E and H (apply Proposition 2.5 (i) or see [ER2, corollary 4.5] or [Pi2, part 3, example 2]). Therefore $C_1(E,F) = \Gamma_2(E_r, F_r)$. Thus taking into account Remark 4.6 and (2.6), we obtain the result by applying Proposition 4.2 with $X = E_r, Y = F_c^*$. ∎

In what follows, we wish to exploit a well-known extension property of representations. Let A, B be C^*-algebras and let $\pi\colon A \to B(K)$ be a representation. Assume that $A \subset B$ (as a C^*-algebraic embedding). Then there exists a c.c. map $\widehat{\pi}\colon B \to B(K)$ extending π. Let us consider C^*-algebraic embeddings $A_i \subset B_i$ $(1 \leq i \leq n)$. Let E, F be Hilbert spaces. From the previous point, an obvious consequence of Christensen and Sinclair's theorem is that any c.b. multilinear map $T\colon A_n \times \cdots \times A_1 \to B(E,F)$ admits a c.b. extension

$\widehat{T}$: $B_n \times \cdots \times B_1 \to B(E,F)$ with $\|\widehat{T}\|_{cb} = \|T\|_{cb}$. Of course, our Theorem 1.1 has an analogous consequence in the framework of module c.b. multilinear maps. Before stating this, we recall that Sublemma 3.3 yields an embedding $H^\infty(B(E,F)) \subset B(H^2(E), H^2(F))$ which defines a natural operator space structure on $H^\infty(B(E,F))$ (see the proof of Lemma 3.4). Thus we have:

COROLLARY 4.8: *Let T: $A_n \times \cdots \times A_1 \to H^\infty(B(E,F))$ be a c.b. multilinear map. Then there exists a c.b. multilinear map $\widehat{T}$: $B_n \times \cdots \times B_1 \to H^\infty(B(E,F))$ extending T with $\|\widehat{T}\|_{cb} = \|T\|_{cb}$.*

Remark 4.9: (i) The link between analytic factorizations and extension properties was first noticed in [HP, cor. 2.9] where Corollary 4.8 was proved in the particular case when $n = 1$, $E = F = \mathbb{C}$.

(ii) In the same manner, we can deduce from Theorem 3.1 an extension theorem. Namely, let us consider two C^*-algebras $A \subset B$ and two operator spaces X, Y. Then every bounded map u: $Y \overset{h}{\otimes} A \overset{h}{\otimes} X \to H^\infty$ has an extension $\widehat{u}$: $Y \overset{h}{\otimes} B \overset{h}{\otimes} X \to H^\infty$ with $\|u\| = \|\hat{u}\|$. ∎

We now recall a definition from [HP]. Let X, Y be Banach spaces. Assume that we are given a surjection q: $X \to Y$. We denote by Q: $H^\infty(X) \to H^\infty(Y)$ the natural map associated to q (i.e.: $Q(\varphi)(z) = q(\varphi(z))$). We say that q is an H^∞-surjection (resp. metric H^∞-surjection) if Q is a surjection (resp. a metric surjection). In the following, we keep the notation of Corollary 4.8. We denote by q: $CB(B_n \times \cdots \times B_1, B(E,F)) \to CB(A_n \times \cdots \times A_1, B(E,F))$ the canonical map defined by $q(T) = T_{/A_n \times \cdots \times A_1}$. It follows from the discussion above Corollary 4.8 that q is a metric surjection. Now we can restate Corollary 4.8 as follows:

COROLLARY 4.10: *The canonical restriction map*

$$q: CB(B_n \times \cdots \times B_1, B(E,F)) \to CB(A_n \times \cdots \times A_1, B(E,F))$$

is a metric H^∞-surjection.

Remark 4.11: Given operator spaces X, Y and a map q: $X \to Y$, it is tempting to say that q is an H^∞-complete surjection (resp. metric H^∞-complete surjection) if the maps $q^{(n)}$: $\mathcal{M}_n(X) \to \mathcal{M}_n(Y)$ are uniformly H^∞-surjections (resp. metric H^∞-surjections). We recall that:

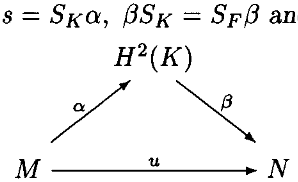
$$\mathcal{M}_n(CB(A_n \times \cdots \times A_1, B(E,F))) = CB(A_n \times \cdots \times A_1, \mathcal{M}_n(B(E,F))).$$

Since $\mathcal{M}_n(B(E,F)) = B(\ell_n^2(E), \ell_n^2(F))$, Corollary 4.8 clearly implies that the canonical map $q: CB(B_n \times \cdots \times B_1, B(E,F)) \to CB(A_n \times \cdots \times A_1, B(E,F))$ is actually a metric H^∞-complete surjection. ∎

Remark 4.12: We would like to mention a simple counter-example about tempting generalizations of Corollary 4.10. Let B be a C^*-algebra such that $L^\infty(\mathbb{T}) \overset{h}{\otimes} L^\infty(\mathbb{T}) \subset B$ (as an operator space embedding). Then the canonical surjection $q: B^* \to (L^\infty(\mathbb{T}) \overset{h}{\otimes} L^\infty(\mathbb{T}))^*$ is not an H^∞-surjection. Indeed, let us assume that q is an H^∞-surjection. By results of [HP], this implies that $(L^\infty(\mathbb{T}) \overset{h}{\otimes} L^\infty(\mathbb{T}))^*$ has the analytic Radon–Nikodym property (see [HP] for the definition and references). It is not hard to deduce from Grothendieck's theorem (see [Pi3, th.5.19] for example) that the injective tensor product $L^1(\mathbb{T}) \overset{\vee}{\otimes} L^1(\mathbb{T})$ is isomorphic to a subspace of $(L^\infty(\mathbb{T}) \overset{h}{\otimes} L^\infty(\mathbb{T}))^*$. Since $L^1(\mathbb{T})$ contains ℓ^2, the space $L^1(\mathbb{T}) \overset{\vee}{\otimes} L^1(\mathbb{T})$ contains c_0. As c_0 does not have the analytic Radon-Nikodym property, we get a contradiction. ∎

Until now, we made crucial use of the operator space structures. However, it is interesting to apply the method leading to Lemma 3.5 and Proposition 4.2 in the framework of Banach spaces. We recall that given two Banach spaces M, N, and a linear map $u: M \to N$, we say that u factors through Hilbert space provided that there are a Hilbert space H and two linear maps $\alpha: M \to H, \beta: H \to N$ such that $u = \beta \circ \alpha$. To avoid confusion with the norm γ_2 defined in section 2, we denote by γ_2^B the norm which corresponds to this factorization. Namely, $\gamma_2^B(u) = \inf \{\|\alpha\| \, \|\beta\|\}$ where the infimum runs over all possible factorizations. We refer to ([Pi3, chap.2]) for informations. Our method allows us to prove a slight improvement of theorems 3.1 and 3.4 in [L]:

PROPOSITION 4.13: *Let M, F be Banach spaces. Let $1 \le p \le +\infty$. Let $N \subset H^p(F)$ be a shift-invariant subspace. Let $s: M \to M$ be a contraction. Let $u: M \to N$ such that $us = S_F u$. Assume that u factors through Hilbert space. Then there exist a Hilbert space K and two operators $\alpha: M \to H^2(K), \beta: H^2(K) \to N$, such that $u = \beta\alpha$, $\alpha s = S_K \alpha$, $\beta S_K = S_F \beta$ and $\|\alpha\|\|\beta\| \le \gamma_2^B(u)$.*



Proof: Left to the reader. ∎

Remark 4.14: In the preceding statement, β may be viewed as a module map from $H^2(E)$ into $H^p(F)$. It is not hard to show that this implies $p \in [1,2]$ or $\beta = 0$. With the notation of Proposition 4.13, we then have the following corollary: for any $p \in]2, +\infty]$ and for any u: $M \to H^p(F)$ such that $us = S_F u$, u factors through Hilbert space iff $u = 0$. ∎

5. Connections with the gamma-norms

This section is devoted to the study of Pisier's gamma norms in the framework of c.b. maps. In their paper [BP], Blecher and Paulsen discovered a very natural link between gamma-norms and Haagerup norms. We will use that (see Lemma 5.3) to show that Pisier's analytic factorization theorem about gamma-norms (see Theorem 5.1 below) may be viewed as a restatement of our Proposition 4.2.

We first recall the necessary definitions about gamma-norms. We follow [Pi1, p.83]. Let X be a Banach space. Let $I(X)$ be a subset of the set of all finite families in X. We assume that there exist two constants $c > 0$ and $C > 0$ for which:

(5.1) $$\forall x \in X, \quad \|x\| \leq c \Longrightarrow \{x\} \in I(X),$$

(5.2) $$\forall \varphi \in X^*, \ \forall \{x_1, \ldots, x_n\} \in I(X), \quad \sum_{i=1}^{n} |\varphi(x_i)|^2 \leq C^2 \|\varphi\|^2.$$

Let H be a Hilbert space. Let A: $X \to H$ be a linear map. We set:

(5.3) $$\delta_1(A) = \text{ Sup } \{(\sum_{i=1}^{n} \|Ax_i\|^2)^{1/2} \mid \{x_1, \ldots, x_n\} \in I(X)\}.$$

We denote by $D_1(X, H)$ the set of all A: $X \to H$ such that $\delta_1(A) < +\infty$ and by (5.1), $(D_1(X, H), \delta_1)$ is a Banach space.

Let us now consider another Banach space Y. We give ourselves a similar set $I(Y)$ and we define δ_2 and D_2 analogously, using $I(Y)$ instead of $I(X)$. We now introduce the space $\Gamma(X, Y^*)$ of all operators T: $X \to Y^*$ for which there exist a Hilbert space H and operators $A \in D_1(X, H)$, $B \in D_2(Y, H^*)$ such that $T = B^* A$. We then define $\gamma(T) = \inf\{\delta_1(A)\delta_2(B)\}$ where the infimum runs over all possible factorizations. It is easily checked that $(\Gamma(X, Y^*), \gamma)$ is a Banach space. The norm γ is called the gamma-norm associated to (δ_1, δ_2). We refer to [Pi1] for various examples of gamma-norms. With the above notation, we have:

THEOREM 5.1 ([Pi1, th 2.3]): *Let* $\varphi \in H^\infty(\Gamma(X,Y^*))$. *Then there exist a Hilbert space* H, $\varphi_1 \in H^\infty(D_1(X,H))$ *and* $\varphi_2 \in H^\infty(D_2(Y,H^*))$ *such that* $\|\varphi_1\|_\infty \|\varphi_2\|_\infty \le \|\varphi\|_\infty$ *and:* $\forall\, z \in D, \quad \varphi(z) = \varphi_2(z)^* \circ \varphi_1(z)$.

The proof of Theorem 5.1 is an obvious combination of Proposition 4.2 with the following:

PROPOSITION 5.2: *There exist operator spaces* X_1 , Y_1 *such that* X_1 *and* X *(resp.* Y_1 *and* Y *) are isomorphic Banach spaces and for any Hilbert space* H, *we have:*

$$D_1(X,H) = CB(X_1, H_c), \tag{5.4}$$

$$D_2(Y,H^*) = CB(Y_1, H_r^*), \tag{5.5}$$

under the natural identifications given by those isomorphisms.

The proof of this proposition relies upon:

LEMMA 5.3:

(i) *Let* Z *be a Banach space. Let* $\mathcal{K}$ *be a set of positive sesquilinear forms on* Z *such that:* $\forall z \in Z$, $\|z\| = \ \mathrm{Sup}\{(\theta(z,z))^{\frac{1}{2}} \mid \theta \in \mathcal{K}\}$. *Then* Z *may be endowed with an operator space structure for which:*
(5.6)
$$\forall z_1 \in Z, \dots, z_i \in Z, \quad \left\| \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \right\|_{\mathcal{M}_{n1}(Z)} = \mathrm{Sup}\Big\{ \Big(\sum_{i=1}^n \theta(z_i,z_i)\Big)^{\frac{1}{2}} \mid \theta \in \mathcal{K} \Big\}.$$

(ii) *Assume that there is a Banach space* X_1 *such that* $X_1^* = Z$ *and* $\mathcal{K} \subset X_1 \otimes \overline{X_1}$. *Then part (i) can be obtained so that* Z *is the standard dual of an operator space structure on* X_1.

Proof: Part (i) was first proved in [BP, p.279-281]. A short proof is given in [Pi5, prop. 4.8]. Using the construction given in this latter proof, it is not hard to obtain (ii). The details are left to the reader. ∎

Proof of Proposition 5.2: We only sketch it. Let us introduce:

$$\mathcal{K} = \Big\{ \sum_{i=1}^n x_i \otimes \overline{x_i} \,|\, \{x_1, \dots x_n\} \in I(X) \Big\} \subset X \otimes \overline{X}.$$

For any $\varphi \in X^*$, we set $|\varphi| = \sup\{\ (\theta(\varphi,\varphi))^{\frac{1}{2}} |\ \theta \in \mathcal{K}\ \}$. The assumptions (5.1) and (5.2) ensure that $|\,.\,|$ is a well-defined equivalent norm on X^*. We now let $Z = (X^*, |\,.\,|)$ the resulting Banach space. Actually Z is the dual space of some Banach space X_1 isomorphic to X. Now we may regard $\mathcal{K}$ as a subset of $X_1 \otimes \overline{X_1}$ and apply Lemma 5.3. It is then easy to deduce (5.4) from (5.6). Of course, the proof of (5.5) is entirely similar. ∎

Remark 5.4: We keep the above notation. Let us consider the set $\widehat{I}(X)$ of finite families in X defined by:

$$\{x_1, \dots, x_n\} \in \widehat{I}(X) \Leftrightarrow \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_{\mathcal{M}_{n1}(X_1)} \leq 1.$$

Then Proposition 2.5 (i) and Proposition 5.2 imply that for any Hilbert space H and any linear map $A: X \to H$,

$$\delta_1(A) = \ \mathrm{Sup}\ \{(\sum_{i=1}^{n} \|Ax_i\|^2)^{1/2} |\ \{x_1, \dots, x_n\} \in \widehat{I}(X)\}.$$

By (5.3), this means that if we replace $I(X)$ by $\widehat{I}(X)$, we get the same norm δ_1. However, $I(X)$ and $\widehat{I}(X)$ may be different.

Example: Let us take for $I(X)$ the set of all $\{x_1, \dots, x_n\}$ such that $\sum_{i=1}^n \|x_i\|^2 \leq 1$, or the set of all $\{x\}$ such that $\|x\| \leq 1$. Then we have $D_1(X,H) = B(X,H)$. Thus if we denote by $(e_i)_{i\geq 1}$ the canonical basis of ℓ^2, we obtain:

$$\{x_1, \dots, x_n\} \in \widehat{I}(X) \Leftrightarrow \Pi_2(\sum_{i=1}^{n} x_i \otimes e_i) \leq 1,$$

where Π_2 denotes the 2-summing norm (see [Pi3] for the definition and necessary information).

Remark 5.5: We proved that our Proposition 4.2 implies Theorem 5.1. Conversely, Proposition 4.2 may be viewed as a corollary of Theorem 5.1. Indeed, let us consider two operator spaces $X \subset B(H), Y \subset B(K)$. Let us define $I(X)$ and $I(Y)$ by :

$$\{x_1, \dots, x_n\} \in I(X) \Leftrightarrow \left\| \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_{\mathcal{M}_{n1}(X)} \leq 1,$$

$$\{y_1, \dots, y_n\} \in I(Y) \Leftrightarrow \|(y_1, \dots, y_n)\|_{\mathcal{M}_{1n}(Y)} \leq 1.$$

Let us define δ_1 by (5.3) and δ_2 similarly. Then by Proposition 2.5, $D_1(X, H) = CB(X, H_c)$ and $D_2(Y, H_r^*) = CB(Y, H_r^*)$. Hence Proposition 4.2 follows from Theorem 5.1. ∎

We end this section by a last improvement of Theorem 1.1 which is closely related to the previous results. Let E, F be Banach spaces. We may define matrix norms on $B(E, F)$ by letting $\mathcal{M}_n(B(E,F)) = B(\ell_n^2(E), \ell_n^2(F))$. It should be noticed that these norms do not satisfy (2.1) and (2.2) (except for the case when E, F are Hilbert spaces). However, we may obviously define c.b. maps valued in $B(E, F)$. Let us now recall Pisier's theorem [Pi2] about this notion :

Let $G \subset B(H)$ be an operator space. For any $u\colon g \to B(E,F)$, $\|u\|_{cb} \leq 1$ iff there are a Hilbert space K, a representation $\pi\colon B(H) \to B(K)$ and two contractions $\alpha\colon E \to K$, $\beta\colon K \to F$ such that:

$$\forall g \in G, \quad u(g) = \beta\, \pi(g)\, \alpha.$$

In other words, the Wittstock–Stinespring theorem extends to this setting with no change. By an obvious reiteration, we can deduce that Theorem 2.1 also remains valid when E and F are Banach spaces. Our last result is:

THEOREM 5.6: *The statement of Theorem 1.1 holds when E, F are merely Banach spaces.*

Proof: We just consider the case $n = 1$ and leave the general case to the reader. We give ourselves a C^*-algebra A and two Banach spaces E, F.

Let us first assume that F is a dual space, we let $F = Y^*$. It is clear from above that E and Y may be equipped with operator space structures such that for any Hilbert space H, $B(E, H) = CB(E, H_c)$ and $B(Y, H^*) = CB(Y, H_r^*)$ (see the example in Remark 5.4). Hence Proposition 2.2 and Pisier's theorem lead us to the identification $CB(A, B(E, F)) = (Y \overset{h}{\otimes} A \overset{h}{\otimes} E)^*$. It is now easy to check that applying Theorem 3.1 to $(Y \overset{h}{\otimes} A \overset{h}{\otimes} E)^*$ gives us the expected result on $CB(A, B(E, F))$. The general case may be easily deduced from above, using a similar argument as in Remark 4.5. ∎

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